The Twistor Transform of a Verlinde formula

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Introduction

Let Σ be a compact Riemann surface of genus g. The moduli space $\mathcal{M}_g = \mathcal{M}_g(2,1)$ of stable rank 2 holomorphic bundles over Σ with fixed determinant bundle of degree 1 is a smooth complex $(3g-3)$-dimensional manifold [25]. The anticanonical bundle of M is the square of a holomorphic line bundle L, some power of which embeds $\mathcal{M}_g$ into a projective space. The dimensions of the vector spaces $H^0(\mathcal{M}_g, \mathcal{O}(L^{m-1}))$ of holomorphic sections of powers of L are known to be independent of the choice of complex structure on Σ, and are given by the formula

$$h^0(\mathcal{M}_g, \mathcal{O}(L^{m-1})) = -m^{g-1} \sum_{i=1}^{2m-1} (-1)^i \mathrm{cosec}^{2g-2}(\frac{i\pi}{2m}) \tag{0.1}$$

predicted by Verlinde [29]. This is closely related to the structure of the cohomology ring of $\mathcal{M}_g$, and a number of independent proofs and generalizations of (0.1) are now known. Below we shall follow closely the approach of Szenes [27].

In the case in which Σ is a hyperelliptic surface, and is therefore a 2-fold branched covering of $\mathbb{CP}^1$, Desale and Ramanan [9] exhibit $\mathcal{M}_g$ as a complex submanifold of the flag manifold $\mathcal{F}_g = SO(2g+2)/(U(g-1)\times SO(4))$. As explained in [27] this reduces verification of (0.1) to certain $SO(2g+2)$-equivariant calculations. Our contribution is to observe that $\mathcal{F}_g$ is the twistor space of the real oriented Grassmannian $\mathcal{G}_g = SO(2g+2)/(SO(2g-2)\times SO(4))$ in the sense of [7, 8] for all $g \geq 3$. This enables us to relate the cohomology of the symmetric space $\mathcal{G}_g$ directly to the cohomology of $\mathcal{M}_g$, and we obtain a set of generators for the latter which may be compared to the universal ones described in [21, 28, 10]. As a feasibility study, we illustrate the theory in the present paper for the case $g = 3$ which is worthy of special attention since the fibration $\mathcal{F}_3 \to \mathcal{G}_3$ encapsulates the quaternionic structure of the base space in a manner first identified by Wolf [31].

In the first section we investigate the cohomology of $\mathcal{G}_3 = SO(8)/(SO(4)\times SO(4))$. Using its quaternionic spin structure, we prove that the odd Pontrjagin classes of $\mathcal{G}_3$ vanish, and that its $\hat{A}$ class simplifies remarkably. In the second section we recover Ramanan's description [23] of the Chern ring of $\mathcal{M}_3$ in the context of the natural mapping $\mathcal{M}_3 \to \mathcal{G}_3$, enabling $h^0(\mathcal{M}_3, L^{m-1})$ to be computed rapidly. Whilst this provides only a particularly simple instance of (0.1), results of the third section identify $H^0(\mathcal{M}_3, L^k)$ with a virtual representation of $SO(8)$ that also arises from the kernels of coupled Dirac operators on $\mathcal{G}_3$. Similar techniques can in theory be applied to higher genus cases, and formulae such as ${p_1}^g = 0$ on $\mathcal{M}_g$ [28, 16] may be expected to interact with properties of $\mathcal{G}_g$ such as the constancy of the elliptic genera considered in [30, 15].

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1. Grassmannian cohomology

From now on we denote by $\mathcal{G}$ the Grassmannian

$$\mathcal{G}_3 = \frac{SO(8)}{SO(4)\times SO(4)} \tag{0.1}$$

that parametrizes real oriented 4-dimensional subspaces of $\mathbb{R}^8$. Let W denote the tautological real rank 4 vector bundle over $\mathcal{G}$, and $W^\perp$ its orthogonal complement in the trivial bundle over $\mathcal{G}$ with fibre $\mathbb{R}^8$. The bundles W and $W^\perp$ arise from the standard representations of the two $SO(4)$ factors constituting the isotropy subgroup in (0.1), and it follows that

$$T\mathcal{G} \cong W\otimes W^\perp. \tag{0.2}$$

The $SO(8)$-invariant Riemannian metric on $\mathcal{G}$ determines an isomorphism $W\cong W^*$ of vector bundles.

The decomposition (0.2) may be refined by lifting the $SO(4)$ structure of W to $Spin(4)\cong SU(2)\times SU(2)$ on a suitable open dense subset $\mathcal{G}'$ of $\mathcal{G}$. This procedure is one that is familiar from the study of Riemannian 4-manifolds, and

$$W_\mathbb{C} \cong U\otimes_\mathbb{C} V,$$

where U and V are each complex rank 2 vector bundles over $\mathcal{G}'$. The resulting isomorphism

$$(T\mathcal{G})_\mathbb{C} \cong U\otimes (V\otimes W_\mathbb{C}^\perp) \tag{0.3}$$

reflects the fact that $\mathcal{G}$ is a quaternion-Kähler manifold [31, 24]. In (0.3), U may be thought of as a quaternionic line bundle (usually called H), and its cofactor $V\otimes W_\mathbb{C}^\perp$ (usually called E) has structure group $SU(2)\times SO(4)$ extending to $Sp(4)$.

The Betti numbers of a quaternion-Kähler $4n$-manifold of positive scalar curvature satisfy $b_{2k+1}=0$ for all k and $b_{2k-4}\le b_{2k}$ for $k\le n+1$. They are also subject to the linear constraint of [17] which for $n=4$ takes the form

$$3(b_2+b_4) = 1+b_6+2b_8.$$

This is well illustrated by $\mathcal{G}$, which has Poincaré polynomial

$$P_t(\mathcal{G}) = 1+3t^4+4t^8+3t^{12}+t^{16},$$

and is the only real Grassmannian to have $b_4>2$. (These facts may be deduced from [12, chapter XI].) We shall in fact only be concerned with the subring generated by the Euler class $e=e(W)$ and the first Pontrjagin class $f=p_1(W)$.

Although the classes e and f are very natural, it will ultimately be more convenient to consider

$$u=-c_2(U),\quad v=-c_2(V).$$

Because of the $\mathbb{Z}_2$-ambiguity in the definition of U,V, the classes u,v are not integral, but the symmetric products $\bigodot^2 U, \bigodot^2 V$ are globally defined so $4u,4v$ belong to $H^4(\mathcal{G},\mathbb{Z})$. If we write formally $4u=\ell^2$ then

$$\mathrm{ch}(U) = e^{\ell/2}+e^{-\ell/2} = 2+u+\frac{1}{12}u^2+\frac{1}{360}u^3+\frac{1}{20160}u^4. \tag{0.4}$$

The class ℓ is given geometrical significance by the splitting (0.2). An analogous expression to (0.4) holds for $\mathrm{ch}(V)$, and from $\mathrm{ch}(W_{\mathbb{C}})=\mathrm{ch}(U)\mathrm{ch}(V)$, we obtain

$$\begin{aligned} e &= u-v,\\ f &= 2(u+v). \end{aligned} \tag{0.5}$$

We may add that $p_2(W)=c_4(W_{\mathbb{C}})=(u-v)^2$ confirming the well-known relation

$$p_2(W)=e^2. \tag{0.6}$$

Moreover, the space $H^4(\mathcal{G},\mathbb{Z})$ is generated by e,f together with $e(W^{\perp})$ [19].

1.1 Proposition *Evaluation on the fundamental cycle* $[\mathcal{G}]$ *yields*

$$\begin{gathered} e^4=2=e^2f^2,\quad e^3f=0=ef^3,\quad f^4=4;\\ u^4=\frac{21}{64}=v^4,\quad u^3v=-\frac{7}{64}=uv^3,\quad u^2v^2=\frac{5}{64}. \end{gathered}$$

We shall deduce these Schubert-type relations from a description of the total Pontrjagin class and the $\hat{A}$ class

$$\begin{aligned} P(T\mathcal{G}) &= 1+P_1+P_2+P_3+P_4,\\ \hat{A}(T\mathcal{G}) &= 1+\hat{A}_1+\hat{A}_2+\hat{A}_3+\hat{A}_4 \end{aligned}$$

of the tangent bundle (0.2) of $\mathcal{G}$. (Upper case P_i's are used to prevent a future clash of notation.) The classes $\hat{A}_i$, $1\le i\le 4$ are determined in terms of the P_i in the usual way [14], and

1.2 Proposition $P_1=0=P_3$ *and* $\hat{A}(\mathcal{G})=1-\dfrac{1}{240}f^2$.

Proof of both propositions. It is easy to check that, in the presence of (0.5), the two sets of equations of Proposition 1.1 are equivalent. The equalities $u^4=v^4$ and $u^3v=uv^3$ are immediate from the symmetry between U and V, and these are equivalent to $ef^3=0=ef^3$. Using (0.6), we have

$$\mathrm{ch}(W_{\mathbb{C}})=4+f+\frac{1}{12}(-2e^2+f^2)+\frac{1}{360}(-3e^2f+f^3)+\frac{1}{20160}(2e^4-4e^2f^2+f^4). \tag{0.7}$$

From (0.2) and (0.7),

$$\begin{aligned} \mathrm{ch}(T\mathcal{G})_{\mathbb{C}} &= (\mathrm{ch}\,W_{\mathbb{C}})(8-\mathrm{ch}W_{\mathbb{C}})\\ &= 16-f^2+\frac{1}{6}(2e^2f-f^3)+\frac{1}{720}(-20e^4+32e^2f^2-9f^4). \end{aligned} \tag{0.8}$$

In particular $P_1=0$, and so we also have

$$\mathrm{ch}(T\mathcal{G})_{\mathbb{C}}=16-\frac{1}{6}P_2+\frac{1}{120}P_3+\frac{1}{10080}(P_2^2-2P_4). \tag{0.9}$$

Comparing (0.8) and (0.9) gives

$$P_2=6f^2,\quad P_3=20(2e^2f-f^3),\quad P_4=140e^4-224e^2f^2+81f^4. \tag{0.10}$$

The remainder of the proof is based on the following less obvious facts.

(i) $\mathcal{G}$ is a spin manifold (see forward to (0.1)) carrying a metric of positive scalar curvature. Therefore its $\hat{A}$ genus

$$\hat{A}_4 = \frac{1}{2^{16}3^4 5^2 7}(762P_1^4 - 1808P_1^2P_2 + 416P_2^2 + 1024P_1P_3 - 384P_4) \tag{0.11}$$

vanishes. Thus

$$0 = 416(6f)^2 - 384(140e^4 - 224e^2f^2 + 81f^4) = 5376(-10e^4 + 16e^2f^2 - 3f^4). \tag{0.12}$$

(ii) The dimension d of the isometry group of any quaternion-Kähler 16-manifold with positive scalar curvature is given by

$$d = 7 - \frac{8}{3}P_1u^3 + 64u^4$$

[24, page 170]. In the present case, $d = \dim SO(8) = 28$ and we obtain

$$21 = 64u^4 = \frac{1}{4}(16e^4 + 24e^2f^2 + f^4). \tag{0.13}$$

(iii) On any compact quaternion-Kähler $4n$-manifold M with positive scalar curvature and $n > 2$, the index

$$\hat{A}(M, \bigodot^2 U) = \left\langle \mathrm{ch}(\bigodot^2 U)\hat{A}, [M] \right\rangle,$$

vanishes; this is a consequence of [24, Corollary 6.7] which is explained in [17]. Given that

$$\mathrm{ch}(\bigodot^2 U) = 3 + 4u + \frac{4}{3}u^2 + \frac{8}{45}u^3 + \frac{4}{315}u^4,$$

$$\hat{A} = 1 - \frac{1}{24}P_1 - \frac{1}{2^5 3^2 5}P_2 - \frac{1}{2^6 3^3 5^1 7}P_3 = 1 - \frac{1}{240}f^2 + \frac{1}{1008}(2e^2f - f^3),$$

and $u = (2e+f)/4$, it follows that

$$24e^4 - 26e^2f^2 + f^4 = 0. \tag{0.14}$$

Proposition 1.1 now follows from (0.12),(0.13),(0.14), and it only remains to prove that $P_3 = 0$. Because of the symmetry between W and $W^\perp$, it suffices to prove that $P_3e = 0 = P_3f$, but this follows from (0.10) and Proposition 1.1. QED

Remark. The vanishing of $\hat{A}_4$ and (0.12) above is in fact equivalent to the vanishing of the index $\hat{A}(M,T)$ of the Dirac operator coupled to the tangent bundle (see (0.2)), essentially the so-called Rarita-Schwinger operator. This index is known to be equivariantly constant on any spin manifold with S^1 action [30], and always vanishes in the homogeneous setting [15].

2. The flag manifold and moduli space

We denote by $\mathcal{F}$ the complex 9-dimensional homogeneous space

$$\mathcal{F}_3 = \frac{SO(8)}{U(2)\times SO(4)} \tag{0.1}$$

that parametrizes complex 2-dimensional subspaces Π of $\mathbb{C}^8$ that are isotropic with respect to a standard $SO(8)$-invariant bilinear form. It has a complex contact structure that was studied in [31] and exhibits it as the twistor space of $\mathcal{G}$ in the sense of [24]. Projecting Π to a real 4-dimensional subspace of $\mathbb{R}^8$ determines an $SO(8)$-equivariant mapping $\pi\colon\mathcal{F}\to\mathcal{G}$, and each fibre of π is isomorphic to $SO(4)/U(2)$ and defines a rational curve in the complex manifold $\mathcal{F}$.

From standard facts about twistor spaces [6, 24, 22], one knows that $\mathrm{Pic}(\mathcal{F})$ is generated by a holomorphic line bundle L on $\mathcal{F}$ such that

(i) the restriction of L to each fibre $\pi^{-1}(x)\cong\mathbb{CP}^{1}$ equals $\mathcal{O}(2)$;

(ii) L^5 is isomorphic to the anticanonical bundle $\mathcal{K}^{-1}$ of $\mathcal{F}$.

The line bundle L admits a square root over an open set $\mathcal{G}'$ of $\mathcal{G}$ on which U and V are defined, there is a C^∞ isomorphism

$$\pi^*U\cong L^{1/2}\oplus L^{-1/2}. \tag{0.2}$$

Let ℓ denote the fundamental class $c_1(L)$ in $H^2(\mathcal{F},\mathbb{Z})$. From the Leray-Hirsch theorem, there is an identity $(\ell/2)^2+\pi^*c_2(U)=0$ of real cohomology classes. In terms of integral classes, and omitting π^*,

$$\ell^2=4u. \tag{0.3}$$

In the notation of the Introduction, let $\mathcal{M}=\mathcal{M}_3$. Szenes exhibits the latter as the zero set of a non-degenerate holomorphic section $s\in H^0(\mathcal{F},\mathcal{O}(\sigma^*))$, where $\sigma=\bigodot^2\tau$ and τ denotes the tautological rank 2 complex vector bundle acquired from the embedding $\mathcal{F}\subset\mathbb{G}r_2(\mathbb{C}^8)$. (Such a section s corresponds to a quadratic form on $\mathbb{C}^8$, but we shall not mention this again until the end of Section 3.) From the coset description (0.1), it follows that

$$\tau\cong L^{-1/2}\otimes\pi^*V; \tag{0.4}$$

the right-hand side is well defined on $\mathcal{F}$, even though the individual factors only make sense locally (for example on $\pi^{-1}(\mathcal{G}')$). Since $V\cong V^*$, we have $\sigma^*\cong L\otimes\pi^*\bigodot^2V$. The resulting holomorphic structure on $\pi^*\bigodot^2V$ coincides with that induced in a standard way from the fact that $\bigodot^2V$ has a self-dual connection on the quaternion-Kähler manifold $\mathcal{G}$, in the sense of [18]. In particular, $\pi^*\bigodot^2V$ is trivial over each fibre $\pi^{-1}\cong\mathbb{CP}^{1}$. From now on we shall write $\bigodot^2V$ in place of $\pi^*\bigodot^2V$, and often omit tensor product signs.

The cohomology classes ℓ,u,v may be pulled back from both $\mathcal{G}$ and $\mathcal{F}$ to $\mathcal{M}$, and we shall denote the resulting elements of $H^i(\mathcal{M},\mathbb{R})$ by the same symbols.

2.1 Proposition *On $\mathcal{M}$, $3u^2+10uv+3v^2=0$, and evaluation on $[\mathcal{M}]$ yields*

$$u^3=\frac{7}{2}=-v^3,\quad uv^2=\frac{3}{2}=-u^2v.$$

Proof. The submanifold $\mathcal{M}$ of $\mathcal{F}$ is Poincaré dual to the Euler class $c_3(\sigma^*)$, which is readily computed from the formula $\mathrm{ch}(\sigma^*)=e^\ell\mathrm{ch}(\bigodot^2V)$ (see (0.4)) and equals $4\ell(u-v)$. Then, for example,

$$\langle u^3,[\mathcal{M}]\rangle=\langle u^3\,c_3(\sigma^*),[\mathcal{F}]\rangle=\langle 4\ell(u^4-u^3v),[\mathcal{F}]\rangle=8\,\langle u^4-u^3v,[\mathcal{G}]\rangle=\frac{7}{2},$$

the last equality from Proposition 1.1. The evaluation of u^2v, uv^2 and v^3 follows in exactly the same way.

Since $H^8(\mathcal{M},\mathbb{R})\cong H^4(\mathcal{M},\mathbb{R})$ is 2-dimensional [20], there must be a non-trivial linear relation $au^2+buv+cv^2=0$. The solution $(a=c)/b=3/10$ can be found by multiplying the left-hand side by u and v in turn. QED

The next result gives an independent derivation of the characteristic ring in the context of the twistor fibration $\mathcal{F} \to \mathcal{G}$.

2.2 Proposition *The Chern and Pontrjagin classes of $\mathcal{M}$ are given by*

$$c_1 = 2\ell, \quad c_2 = 4(3u+v), \quad c_3 = 8\ell u, \quad c_4 = -\frac{112}{3}uv, \quad c_5 = c_6 = 0;$$
$$p_1 = -8(u+v), \quad p_2 = \frac{3}{8}{p_1}^2, \quad p_3 = 0.$$

Proof. It is known [24] that the fibration π gives a C^∞ splitting of the holomorphic tangent bundle of $\mathcal{F}$:
$$T^{1,0}\mathcal{F} \cong L \,\oplus\, L^{1/2}(V \otimes W_{\mathbb{C}}^{\perp}).$$
Combining this with the isomorphism
$$T^{1,0}\mathcal{F}|_{\mathcal{M}} \cong T^{1,0}\mathcal{M} \oplus (L \textstyle\bigodot^2 V)|_{\mathcal{M}},$$
we obtain
$$\begin{aligned} \mathrm{ch}(T^{1,0}\mathcal{M}) &= e^\ell + e^{\ell/2}\mathrm{ch}(V\, W_{\mathbb{C}}^{\perp}) - e^\ell \mathrm{ch}(\textstyle\bigodot^2 V) \\ &= e^\ell(1 + e^{-\ell/2}\mathrm{ch}V(8 - \mathrm{ch}W_{\mathbb{C}}) - \mathrm{ch}(\textstyle\bigodot^2 V)). \end{aligned}$$
This yields the required expressions for c_1, c_2, c_3. We also get $c_4 = 28(u+v)^2$ which reduces to $-112uv/3$ from Proposition 2.1. We next obtain $c_5 = -32\ell v(u+v)$, so that $c_5\ell = 0$ and the vanishing of c_5 follows from the fact that $H^2(\mathcal{M},\mathbb{R})$ is 1-dimensional [20]. Finally, all these equalities combine to yield
$$c_6 = \frac{1}{3}(504u^3 + 2824u^2v + 1928uv^2 + 120v^3),$$
and Proposition 2.1 implies that $c_6 = 0$. The Pontrjagin classes p_i of $\mathcal{M}$ are now determined from the Chern classes by the usual relations. QED

Remark. The cohomology ring and Chern classes of $\mathcal{M}$ were computed in [23, Theorem 4], and comparison with that shows that
$$h = \ell, \quad \nu = \frac{1}{2}(3u+v).$$
In general, it is known that the total Pontrjagin class of $\mathcal{M}_g$ equals $(1+\frac{1}{2g-2}p_1)^{2g-2}$ [21]. Moreover, ${p_1}^g = 0$ [16, 28] and $c_i = 0$ if $i > 2g-2$ [11].

The above enable the dimension d_k of $H^0(\mathcal{M},\mathcal{O}(L^k))$ to be computed quickly. For this purpose it is convenient to set $k = m-1$.

2.3 Theorem $d_{m-1} = \dfrac{1}{45}m^2(11 + 20m^2 + 14m^4)$.

Proof. Given that $c_1(T^{1,0}F) = 2\ell$, the Todd class $\mathrm{td}(T^{1,0}\mathcal{M})$ of $\mathcal{M}$ equals
$$e^\ell \hat{A}(T\mathcal{M}) = e^\ell\left[1 - \frac{1}{24}p_1 + \frac{1}{2^7 3^2 5}(7{p_1}^2 - 4p_2)\right].$$

Using Propositions 2.1, 2.2 and the Riemann-Roch theorem, we obtain

$$\begin{aligned} d_{m-1} &= \left\langle e^{m\ell}(1+\frac{1}{3}(u+v)-\frac{11}{135}uv),[\mathcal{M}]\right\rangle \\ &= -\frac{22}{135}m^2u^2v+\frac{2}{9}m^4(u^3+u^2v)+\frac{4}{45}m^6u^3, \end{aligned}$$

and the result follows. QED

3. Equivariant indexes

In this section, we begin by considering the Dirac operator over the Grassmannian $\mathcal{G}$. Recall from (0.3) that the quaternionic structure of $\mathcal{G}$ is characterized by the vector bundles $H = U$ and $E \cong V\, W_{\mathbb{C}}$ (juxtaposition denotes tensor product). For $p \geq 4$, the exterior power $\bigwedge^p E$ contains a proper subbundle $\bigwedge_0^p E$ with the property that $\bigwedge^p E \cong \bigwedge_0^p E \oplus \bigwedge^{p-2} E$ and, as described in [4], the total spin bundle Δ of $\mathcal{G}$ decomposes as $\Delta_+ \oplus \Delta_-$ where

$$\begin{aligned} \Delta_+ &\cong \bigodot^4 U \;\oplus\; \bigodot^2 U \textstyle\bigwedge_0^2 E \;\oplus\; \bigwedge_0^4 E, \\ \Delta_- &\cong \bigodot^3 U\, E \;\oplus\; U \textstyle\bigwedge_0^3 E. \end{aligned} \tag{0.1}$$

The fact that all the summands on the right-hand side are globally defined confirms that $\mathcal{G}$ is spin, though we shall not in fact need the decompositions (0.1).

Now let X be any other complex vector bundle over $\mathcal{G}$. The choice of a connection on X allows one to extend the Dirac operator on $\mathcal{G}$ to an elliptic operator

$$D_X\colon \Gamma(\Delta_+\, X) \longrightarrow \Gamma(\Delta_-\, X).$$

The index of this coupled Dirac operator is by definition $\dim(\ker D_X) - \dim(\operatorname{coker} D_X)$. This extends to a homomorphism $K(\mathcal{G}) \to \mathbb{Z}$, so that the index of D_X is also defined when X is a virtual vector bundle. The Atiyah-Singer index theorem [3] asserts that the index of D_X equals

$$\hat{A}(\mathcal{G}, X) = \left\langle \operatorname{ch}(X)\hat{A}(T\mathcal{G}), [\mathcal{G}] \right\rangle \tag{0.2}$$

In our situation, this fact is closely related to the Riemann-Roch theorem on $\mathcal{F}$ which provides the following interpretation of d_k.

Theorem 3.1 *Let* $X_k = \bigodot^{2k+4}U - \bigodot^{2k+2}U \bigodot^2 V + \bigodot^{2k}U \bigodot^2 V - \bigodot^{2k-2}U$, $k \geq 1$. *Then* $d_k = \hat{A}(\mathcal{G}, X_k)$.

Proof. Let σ denote the rank 3 vector bundle $\bigodot^2 \tau$ as above, and let (k) denote the operation of tensoring with L^k. The description of $\mathcal{M}$ as the zero set of a section of $\sigma^* \cong \bigodot^2 V(1)$ provides a Koszul complex

$$0 \to \mathcal{O}_{\mathcal{F}}(\textstyle\bigwedge^3 \sigma(k)) \to \mathcal{O}_{\mathcal{F}}(\bigwedge^2 \sigma(k)) \to \mathcal{O}_{\mathcal{F}}(\sigma(k)) \to \mathcal{O}_{\mathcal{F}}(k) \to \mathcal{O}_{\mathcal{M}}(k) \to 0,$$

or equivalently,

$$0 \to \mathcal{O}_{\mathcal{F}}(k-3) \to \mathcal{O}_{\mathcal{F}}(\bigodot^2 V(k-2)) \to \mathcal{O}_{\mathcal{F}}(\bigodot^2 V(k-1)) \to \mathcal{O}_{\mathcal{F}}(k) \to \mathcal{O}_{\mathcal{M}}(k) \to 0.$$

It follows that

$$\chi(\mathcal{M}, \mathcal{O}(k)) = a_k - b_{k-1} + b_{k-2} - a_{k-3}, \tag{0.3}$$

where

$$a_k = \chi(\mathcal{F}, \mathcal{O}(k)), \quad b_k = \chi(\mathcal{F}, \mathcal{O}(\bigodot^2 V(k))). \tag{0.4}$$

These holomorphic Euler characteristics may be computed using the Riemann-Roch theorem and the cohomological version [24, 7.2] of the twistor transform; the result is

$$a_k = \hat{A}(\mathcal{G}, \bigodot^{2k+4} U), \quad b_k = \hat{A}(\mathcal{G}, \bigodot^{2k+4} U \bigodot^2 V). \tag{0.5}$$

Finally, Proposition 2.2 implies that the canonical bundle $\mathcal{K}(\mathcal{M})$ is isomorphic to L^{-2}, so by Serre duality and Kodaira vanishing, $H^i(\mathcal{M}, \mathcal{O}(k)) = 0$ for all $i \geq 1$ and $k \geq -1$. In particular, $\chi(\mathcal{M}, \mathcal{O}(k)) = \dim H^0(\mathcal{M}, \mathcal{O}(k))$ for all $k \geq -1$, and the theorem now follows from (0.3). QED

The isometry group $SO(8)$ of $\mathcal{G}$ acts naturally on the cohomology groups over $\mathcal{F}$ of the sheaves $\mathcal{O}(k)$, $\mathcal{O}(\bigodot^2 V(k))$ considered above. The integers a_k, b_k and

$$d_k = a_k - b_{k-1} + b_{k-2} - a_{k-3}$$

are therefore the dimensions of certain virtual $SO(8)$-modules, and we identify these shortly.

Let $V(\gamma)$ denote the complex irreducible representation of $SO(8)$ with dominant weight γ, where $\gamma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0$. We adopt standard coordinates so that $V(1,0,0,0) = \mathbb{C}^8$ is the fundamental representation, and $V(1,1,0,0) = \mathfrak{so}(8, \mathbb{C})$ is the complexified adjoint representation.

3.2 Proposition *Let* $A_k = V(k,k,0,0)$ *and* $B_k = V(k+1, k-1, 0, 0)$*. Then* $a_k = \dim A_k$ *and* $b_k = \dim B_k$.

Proof. The Weyl dimension formula states that

$$\dim(V(\gamma)) = \prod_{\alpha \in R_+} \frac{\langle \alpha, d+\gamma \rangle}{\langle \alpha, d \rangle},$$

where R_+ denotes the set of positive roots and d is half of their sum. With the above coordinates,

$$\begin{aligned} R_+ = \{&(1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1), \\ &(1,-1,0,0), (1,0,-1,0), (1,0,0,-1), (0,1,-1,0), (0,1,0,-1), (0,0,1,-1)\}, \end{aligned}$$

$d = (3,2,1,0)$ and we obtain

$$\begin{aligned} \dim A_k &= \frac{1}{4320}(k+1)(k+2)^3(2k+5)(k+3)^3(k+4), \\ \dim B_k &= \frac{1}{1440}k(k+1)^2(k+2)(2k+5)(k+3)(k+4)^2(k+5). \end{aligned}$$

We claim that the right-hand sides are equal to a_k and b_k respectively. It follows from (0.4) that a_k and b_k are polynomials in k of degree 9, and by Serre duality,

$$a_{-k} = -a_{k-5}, \quad b_{-k} = -b_{k-5}, \qquad k \in \mathbb{Z}. \tag{0.6}$$

By (0.6) and suitable vanishing theorems [5], $a_k = 0 = b_k$ for $k = -4, -3, -\frac{5}{2}, -2, -1$. In addition, $\mathcal{F}$ has Todd genus $a_0 = 1 = -a_{-5}$, and $b_0 = 0 = b_{-5}$. Accordingly,

$$\begin{aligned} a_k &= \frac{1}{4320}(k+1)(k+2)(2k+5)(k+3)(k+4)\tilde{a}_k, \\ b_k &= \frac{1}{1440}k(k+1)(k+2)(2k+5)(k+3)(k+4)(k+5)\tilde{b}_k. \end{aligned}$$

where $\tilde{a}_k$ is a quartic polynomial in k with $\tilde{a}_0 = 36$ and $\tilde{b}_k$ is quadratic in k.

Let $n = 2k+4$. The formulae (0.5) involve $\mathrm{ch}(\bigodot^n U) = f(n)$, where

$$f(x) = \frac{e^{(x+1)\ell/2} - e^{-(x+1)\ell/2}}{e^{\ell/2} - e^{-\ell/2}} \tag{0.7}$$

(see (0.2)). To evade an explicit calculation of $\mathrm{ch}(\bigodot^n U)$, we exploit the following formulae which are easily deduced from (0.7).

3.3 Lemma $f'(0) = \dfrac{\ell/2}{\tanh(\ell/2)}, \quad f''(0) = u$.

The right-hand side of the first equation is the series used in the definition of Hirzebruch's L-genus, and using (0.3) can be rewritten as

$$\begin{aligned} \frac{d}{dn}\Big|_{n=0} \mathrm{ch}(\textstyle\bigodot^n U) &= 1 - \sum_{j\geq 1} (-1)^j \frac{2^{2j} B_j}{(2j)!} u^{2j} \\ &= \frac{1}{2}(1 - \frac{1}{3}u - \frac{1}{45}u^2 + \frac{2}{945}u^3 - \frac{1}{4725}u^4), \end{aligned}$$

where B_j are the Bernoulli numbers [14]. From above, we obtain

$$\frac{d}{dk}\Big|_{k=-2} a_k = \frac{1}{270}(u^4 + 2u^3 v + u^2 v^2) - \frac{1}{4725}u^4 = 0 = \frac{d^2}{dk^2}\Big|_{k=-2} a_k.$$

It follows that $\tilde{a}_k$ is divisible by $(k+2)^2$, and by Serre duality by $(k+3)^2$. We obtain $\tilde{a}_k = (k+2)^2(k+3)^2$. The identification $\tilde{b}_k = (k+1)(k+4)$ is similar, and proceeds using a less-enlightening version of Lemma 3.3; we omit the details. QED

The following table displays some of the above dimension functions in terms of k.

k	0	1	2	3	4	5	6	7	8
a_k	1	28	300	1925	8918	32928	102816	282150	698775
b_k	0	35	567	4312	21840	85050	274890	772464	1945944
d_k	1	28	265	1392	5145	15100	37681	83392	168273

Applying Serre duality and Kodaira vanishing over $\mathcal{F}$, recalling that $\mathcal{K}(\mathcal{F}) \cong L^{-5}$, shows that there is in fact an $SO(8)$-equivariant isomorphism $A_k \cong H^0(\mathcal{F}, \mathcal{O}(k))$. In particular, A_1 may be identified with both the space of holomorphic sections of L and the Lie algebra $\mathfrak{so}(8, \mathbb{C})$ of infinitesimal automorphisms of the contact structure of $\mathcal{F}$. There is an associated moment mapping $\mathcal{F} \to \mathbb{P}(\mathfrak{so}(8, \mathbb{C})^*) \cong \mathbb{CP}^{27}$ that identifies $\mathcal{F}$ with the projectivization of the nilpotent orbit of minimal dimension [26]. Accordingly, the $SO(8)$-equivariant linear mapping

$$\phi_k \colon \bigodot^k (H^0(\mathcal{F}, \mathcal{O}(1))) \longrightarrow H^0(\mathcal{F}, \mathcal{O}(k)) \tag{0.8}$$

is onto for all $k \geq 1$. Indeed, A_k is the irreducible summand of $\bigodot^k A_1$ of highest weight, and it suffices to show that the restriction of ϕ_k to A_k is an isomorphism. Observe that A_k contains a decomposable tensor product $\xi^{\otimes k}$ for some non-zero $\xi \in A_1$ and $\phi_k(\xi^{\otimes k})$, being the kth power of ξ regarded as a section of L, is also non-zero. The irreducibility of A_k and Schur's lemma establishes the claim.

A similar argument can be given to establish an $SO(8)$-equivariant isomorphism $B_k \cong H^0(\mathcal{F}, \mathcal{O}(\bigodot^2 V(k)))$, given that $H^i(\mathcal{F}, \mathcal{O}(\bigodot^2 V(k)))$ vanishes for all $i > 0$ and $k \geq 0$. One considers the mapping

$$\psi_k \colon H^0(\mathcal{F}, \mathcal{O}(\bigodot^2 V(1))) \otimes H^0(\mathcal{F}, \mathcal{O}(k-1)) \longrightarrow H^0(\mathcal{F}, \mathcal{O}(\bigodot^2 V(k))),$$

in which $H^0(\mathcal{F}, \mathcal{O}(\bigodot^2 V(1)))$ is isomorphic to the irreducible 35-dimensional $SO(8)$-module $\bigodot^2_0 \mathbb{C}^8$ with highest weight $(2, 0, 0, 0)$. The irreducible summand of highest weight in the tensor product is isomorphic to B_k and the restriction of ψ_k to this is an isomorphism.

The above arguments can be streamlined by applying more sophisticated twistor transform machinery contained, for example, in [5]. In particular, A_k and B_k are known to be isomorphic to the respective kernels of natural twistor operators

$$\begin{aligned} \alpha_k &\colon \bigodot^{2k} U \longrightarrow E \bigodot^{2k+1} U, \\ \beta_k &\colon \bigodot^{2k} U \bigodot^2 V \longrightarrow E \bigodot^{2k+1} U \bigodot^2 V. \end{aligned}$$

Recall that $\mathcal{M}$ is the zero set of an element s of the space $B_1 \cong \bigodot^2_0 \mathbb{C}^8$. For suitable hyperelliptic surfaces Σ, the section s will be a real element; at each point of $\mathcal{G}$ it then defines a section of $W \oplus W^\perp$, which is a trivial bundle with fibre $\mathbb{R}^8$ (see (0.1)). In these terms the element $\tilde{s} \in \ker \beta_1$ determined by s is essentially the image of s by the homomorphism

$$\bigodot^2 (W \oplus W^\perp)_{\mathbb{C}} \to \bigodot^2 W_{\mathbb{C}} \to \bigodot^2 U \bigodot^2 V \cong \mathrm{Hom}(\bigodot^2 V, \bigodot^2 U).$$

This may be used to describe $\mathcal{M}$ as a 'branched cover' of a real subvariety of $\mathcal{G}$.

The Horrocks instanton bundle over $\mathbb{CP}^5$ discussed at the end of [18] provides an analogous situation in which a geometric object is defined by a non-degenerate solution of a twistor equation over a homogeneous space. Such situations are worthy of more systematic investigation.

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